Quantify Entanglement for Multipartite Quantum States

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In this paper, we consider the problem of how to quantify entanglement for any multipartite quantum states. For bipartite pure states partial entropy is a good entanglement measure. By using partial entropy, we firstly introduce the Combinatorial Entropy of Fully entangled states (CEF) which can be used to quantify entanglement for any fully entangled pure states. In order to quantify entanglement for any multipartite states we also need another concept the Entanglement Combination (EC) which can be used to completely describe the entanglement between any parties of the given quantum states. Combining CEF with EC, we define the Combinatorial Entropy (CE) for any multipartite pure states and present some nice properties which indicate CE is a good entanglement measure. Finally, we point out the feasibility of extending these three concepts to mixed quantum states.

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Quantum entanglement, first noted by Einstein, Podolsky, and Rosen [1] and Schrödinger [2], is one of the essential features of quantum mechanics. Entanglement plays an important role in the theory and application of quantum information and quantum computation [3, 4]. An important problem in quantum computation and information theory is the formulation of appropriate methods for detecting entanglement and then finding measures that quantify the degree of entanglement in multipartite systems. A good measure of entanglement will enhance our understanding of the phenomenon.

The quantification of multipartite entanglement is an open and very challenging problem. An exhaustive definition of bipartite entanglement exists and hinges upon the partial entropy [5, 6], but the problem of defining multipartite entanglement is more difficult [7] and no unique definition exists. Many different measures of entanglement have been proposed which tend indeed to focus on different aspects of the problem, capturing different features of entanglement [8, 9, 10, 11, 12, 13, 14]. For mixed states, the situation is further complicated, even for two qutrits there is no consensus on how to quantify entanglement [15].

In this paper, we present a method to quantify entanglement for any multipartite states by introducing three useful concepts which are the Combinatorial Entropy of Fully entangled pure states (CEF), the Entanglement Combination (EC) and the Combinatorial Entropy (CE) of multipartite pure states. At the same time, we get some nice properties.

For bipartite pure states it has been shown [16, 17, 18] that asymptotically there is only one kind of entanglement and partial entropy is a good entanglement measure for it. We start to look at the partial entropy which is the von Neumann entropy $S(\rho) = -\operatorname{tr}(\rho \log_2 \rho)$ of the reduced density operator obtained by tracing out either of the two parties. Partial entropy has the nice properties that for pure states it is invariant under local unitary transformations (LU) and its expectation does not increase under local operations and classical communication (LOCC).

Consider a n-partite pure state $|\psi\rangle$ in quantum system $H = H^{A_1} \otimes H^{A_2} \otimes \cdots \otimes H^{A_n}$. $P = \{A_1, A_2, \cdots, A_n\}$ is the parties set. Let I denote a nontrivial subset of the parties and let $\bar{I}$ be the set of remainder parties. The n-partite pure state $|\psi\rangle$ can be regard as a bipartite pure state in $H = (\bigotimes_{A_i \in I} H^{A_i}) \otimes (\bigotimes_{A_j \in \bar{I}} H^{A_j})$, denoted by $|\psi_I\rangle$. Then the reduced density operator of subset I of the parties is defined as

$$\rho_I(|\psi\rangle) = \operatorname{tr}_{\bar{I}}(|\psi\rangle\langle\psi|). \tag{1}$$

The partial entropy of subset I is the von Neumann entropy

$$S_I(|\psi\rangle) = -\operatorname{tr}(\rho_I(|\psi\rangle)\log_2 \rho_I(|\psi\rangle)). \tag{2}$$

If the n-partite pure state $|\psi\rangle$ is fully entangled, $|\psi_{I_k}\rangle$ are entangled bipartite pure states for any nontrivial subsets $I_k (k = 1, 2, \cdots, 2^n - 2)$ of P. So we can calculate the partial entropies of subsets I_k by Eq. (2) and $S_{I_k}(|\psi\rangle) > 0$. Summing up the partial entropies $S_{I_k}(|\psi\rangle)$ for all the nontrivial subsets $I_k (k = 1, 2, \cdots, 2^n - 2)$, we get the following definition which can be used to quantify the entanglement of fully entangled pure states.

Definition 1. *Suppose that $|\psi\rangle$ is a fully entangled n-partite pure state in $H = H^{A_1} \otimes H^{A_2} \otimes \cdots H^{A_n}$. $P = \{A_1, A_2, \cdots, A_n\}$ is the parties set. The Combinatorial Entropy of the fully entangled pure state $|\psi\rangle$ can be defined as:*

$$CEF_P(|\psi\rangle) = \begin{cases} 0, & n = 1; \\ \frac{1}{2}\sum_{\emptyset \neq I_k \subsetneq P} S_{I_k}(|\psi\rangle), & n > 1. \end{cases} \tag{3}$$

Where $\frac{1}{2}$ is the normalized factor to make sure that CEF is just the partial entropy for bipartite pure state. $S_{I_k}(|\psi\rangle)$ is the von Neumann entropy defined by Eq. (2).

Now we are ready to show some properties of CEF.

Property 1. *(1) CEF is nonnegative for any fully entangled pure state. $CEF = 0$ if and only if $n = 1$.*
(2) CEF is invariant under LU.

Proof. The first property can be got from Definition 1 directly. Now we prove the second property.

For a given fully entangled pure state $|\psi\rangle$ in quantum system $H = H^{A_1} \otimes H^{A_2} \otimes \cdots \otimes H^{A_n}$ with dimension $d = d^{A_1} \cdot d^{A_2} \cdot \cdots \cdot d^{A_n}$, it can be write in the form $|\psi\rangle = \sum\limits_{i_1=0}^{d_1-1} \sum\limits_{i_2=0}^{d_2-1} \cdots \sum\limits_{i_n=0}^{d_n-1} a_{i_1 i_2 \cdots i_n} |e_{i_1}^{A_1}\rangle |e_{i_2}^{A_2}\rangle \cdots |e_{i_n}^{A_n}\rangle$, where $\{|e_{i_k}^{A_k}\rangle\}_{i_k=0}^{d_k-1}$ are the orthonormal basis of subsystems $H^{A_k} (k = 1, 2, \cdots, n)$. Suppose that U^{A_k} are unitary operators acting on the k-th subsystem H^{A_k} respectively for $k = 1, 2, \cdots, n$. Let

$$|f_{i_k}^{A_k}\rangle = U^{A_k} |e_{i_k}^{A_k}\rangle (k = 1, \cdots, n; i_k = 0, \cdots, d_k - 1), \quad (4)$$

which means that $\{|f_{i_k}^{A_k}\rangle\}_{i_k=0}^{d_k-1}$ is another orthonormal basis of subsystem H^{A_k} for $k = 1, 2, \cdots, n$. We have

$$\begin{aligned} |\phi\rangle &= \bigotimes_{k=1}^{n} U^{A_k} |\psi\rangle \\ &= \sum_{i_1=0}^{d_1-1} \sum_{i_2=0}^{d_2-1} \cdots \sum_{i_n=0}^{d_n-1} a_{i_1 i_2 \cdots i_n} |f_{i_1}^{A_1}\rangle |f_{i_2}^{A_2}\rangle \cdots |f_{i_n}^{A_n}\rangle \quad (5) \end{aligned}$$

Suppose that I is a nontrivial subset of P. Let $I = \{A^1, A^2, \cdots, A^t\}$ without losing the generality. We have

$$\begin{aligned} \rho_I(|\phi\rangle) &= \mathrm{tr}_{\bar{I}} \left(|\phi\rangle\langle\phi| \right) = \sum_{i_{t+1}=0}^{d_{t+1}-1} \cdots \sum_{i_n=0}^{d_n-1} \langle f_{i_{t+1}}^{A_{t+1}} | \cdots \langle f_{i_n}^{A_n} | \phi\rangle\langle\phi| f_{i_{t+1}}^{A_{t+1}}\rangle \cdots | f_{i_n}^{A_n}\rangle \\ &= \bigotimes_{k\in I} U^{A_k} \left(\sum_{i_{t+1}=0}^{d_{t+1}-1} \cdots \sum_{i_n=0}^{d_n-1} \langle e_{i_{t+1}}^{A_{t+1}} | \cdots \langle e_{i_n}^{A_n} | \psi\rangle\langle\psi| e_{i_{t+1}}^{A_{t+1}}\rangle \cdots | e_{i_n}^{A_n}\rangle \right) \bigotimes_{k\in I} \left(U^{A_k}\right)^\dagger = \bigotimes_{k\in I} U^{A_k} \cdot \rho_I(|\psi\rangle) \cdot \bigotimes_{k\in I} \left(U^{A_k}\right)^\dagger \quad (6) \end{aligned}$$

According to Eq. (6), we have

$$\begin{aligned} & S_I\left(|\phi\rangle\right) = -\,\mathrm{tr}\left(\rho_I\left(|\phi\rangle\right) \log_2 \rho_I\left(|\phi\rangle\right)\right) \\ = & -\,\mathrm{tr}\left(\bigotimes_{k\in I} U^{A_k} \cdot \rho_I\left(|\psi\rangle\right) \log_2 \rho_I\left(|\psi\rangle\right) \cdot \bigotimes_{k\in I} \left(U^{A_k}\right)^\dagger \right) \\ = & -\,\mathrm{tr}\left(\rho_I\left(|\psi\rangle\right) \log_2 \rho_I\left(|\psi\rangle\right)\right) = S_I\left(|\psi\rangle\right) \quad (7) \end{aligned}$$

Summing up all the nontrivial subsets I_k and using Eq. (7), we have $CEF\left(|\phi\rangle\right) = CEF\left(|\psi\rangle\right)$. □

These two properties tell us that CEF can be used to quantify the entanglement of any fully entangled pure states. But most of the multipartite pure states are not fully entangled, then how can we quantify the entanglement of them.

For example, consider the 4-qubits pure state $|\psi\rangle = |EPR\rangle \otimes |EPR\rangle$ in $H = \bigotimes_{i=1}^{4} H^{A_i}$, where $|EPR\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is the famous EPR state. Let $I_1 = \{A_1\}, I_3 = \{A_3\}$ and $I_{13} = \{A_1, A_3\}$, then we have $S_{I_{13}}(|\psi\rangle) = S_{I_1}(|\psi\rangle) + S_{I_3}(|\psi\rangle)$ which means that the entanglement between $\{A_1, A_3\}$ and $\{A_2, A_4\}$ can be divided into two parts because $|\psi\rangle$ is partially separable between $\{A_1, A_2\}$ and $\{A_3, A_4\}$. And the partial entropy $S_{I_1}(|\psi\rangle)$ $(S_{I_3}(|\psi\rangle))$ is indeed the entanglement between A_1 and A_2 (A_3 and A_4) which means that we only need to consider the entanglement between fully entangled parties. This example tells us that if we want to quantify the entanglement of multipartite states we should find out all the combinations of fully entangled parties. So we introduce the following concept.

Definition 2. *Suppose that $|\psi\rangle$ is a n-partite pure state in $H = H^{A_1} \otimes H^{A_2} \otimes \cdots H^{A_n}$. $P = \{A_1, A_2, \cdots, A_n\}$ is the parties set. The Entanglement Combination of $|\psi\rangle$ can be defined as:*

$$EC(|\psi\rangle) = \left[(I_1), (I_2), \cdots, (I_r)\right], \quad (8)$$

where $I_k (k = 1, 2, \cdots, r)$ are subsets of P with the following two conditions:

1. *$\bigcup\limits_{k=1}^{r} I_k = P$ and $I_i \cap I_j = \emptyset$ if $i \neq j$;*
2. *For any parties A_{i_a} in I_i and B_{j_b} in I_j, they are entangled if $i = j$ and separable if $i \neq j$.*

Note: We can get the unique definition by giving some rules of the order of $I_k (k = 1, 2, \cdots, r)$ such as in Algorithm 1.

The following properties can be easily got from the definition.

Property 2. *(1) If $r = n$, we have $[(I_1), (I_2), \cdots, (I_r)] = [(A_1), (A_2), \cdots, (A_n)]$ which means that the pure state is separable. If $r = 1$, we have $[(I_1), (I_2), \cdots, (I_r)] = [(A_1, A_2, \cdots, A_n)]$ which means*

that the pure state is fully entangled. If $1 < r < n$, the pure state is partially entangled and $[(I_1), (I_2), \cdots, (I_r)]$ display all the combinations of fully entangled parties.

(2) The parties are entangled if and only if they are in the same combination.

We can use EC to do the qualitative analysis of entanglement for any multipartite quantum states. In order to calculate EC for any given multipartite pure states we need some separability criterions which have been studied in [19] and references therein. Before putting forward an efficient algorithm, we firstly review the following useful lemma [3].

Lemma 1. *A bipartite pure state $|\psi\rangle$ in $H^{A_1} \otimes H^{A_2}$ is separable if and only if $\mathrm{rank}(\rho_{A_1}) = \mathrm{rank}(\rho_{A_2}) = 1$, if and only if ρ_{A_1} and ρ_{A_2} are density operators of pure states. A bipartite pure state $|\psi\rangle$ in $H^{A_1} \otimes H^{A_2}$ is entangled if and only if $\mathrm{rank}(\rho_{A_1}) = \mathrm{rank}(\rho_{A_2}) > 1$, if and only if ρ_{A_1} and ρ_{A_2} are density operators of mixed states.*

By using Lemma 1, we can judge the separability of $|\psi_{I_k}\rangle$ for any nontrivial subset I_k of P. In order to get an efficient algorithm, we need not judge all the separability of $|\psi_{I_k}\rangle$ for $k = 1, 2, \cdots, 2^n - 2$. The main ideas are that (1) if we have already find a fully entangled combination I_k we trace out all parties A_{k_i} in I_k and get a new pure state in a lower dimensional quantum system; (2) we only need to consider the reduced state in the following steps; (3) if we have already put all parties in some combination, the EC of $|\psi\rangle$ is obtained. The algorithm can be constructed as follows:

Algorithm 1. *For any given n-partite pure state $|\psi\rangle$ in $H = H^{A_1} \otimes H^{A_2} \otimes \cdots \otimes H^{A_n}$, let $N = \lceil \frac{n}{2} \rceil - 1$.*

1. *Consider all the combinations with m parties. m ranges from 1 to N.*

   (1) *Denote all the combinations of P with m parties to be $J_k\,(k = 1, 2, \cdots, M)$ where $M = \frac{n!}{m!(n-m)!}$. $\bar{J}_k$ is the complement set of J_k. Judge the separablility of $|\psi_{J_k}\rangle$. If $|\psi_{J_k}\rangle$ is separable go to (2); if $|\psi_{J_k}\rangle$ is entangled, let $k \leftarrow k + 1$ and judge the next until $k = M$.*

   (2) *Let $I_r = J_k$. We obtain the r-th combination of EC. Renew $r \leftarrow r + 1$, trace out all the parties in J_k and get the reduced pure states $|\psi^{'}\rangle$ in $H^{'} = \bigotimes_{A_{k_i} \in \bar{J}_k} H^{A_{k_i}}$ [the reduced state $|\psi^{'}\rangle$ is a pure state which can be ensured by Lemma 1]. Renew $|\psi\rangle \leftarrow |\psi^{'}\rangle, H \leftarrow H^{'}$ and $n \leftarrow n - m$. Go to (1).*

2. *If there are some parties remained, let $r \leftarrow r + 1$ and put all the remained parties in I_r.*

For example, let $|\psi\rangle = \frac{1}{2}(|000000\rangle + |000111\rangle + |110000\rangle + |110111\rangle)$ is a 6-qubits pure state in $H = \bigotimes_{k=1}^{6} H^{A_k}$. We can calculate $EC(|\psi\rangle)$ as follows: (1) For one party combinations, let $J_k = \{A_k\}(k = 1, 2, \cdots, 6)$. We can easily calculate that $\mathrm{rank}(\rho_{A_i}) = 2(i = 1, 2, 4, 5, 6)$ and $\mathrm{rank}(\rho_{A_3}) = 1$ which means that $J_3 = \{A_3\}$ is the first fully entangled combination, so we have $I_1 = \{A_3\}$. Tracing out the party A_3, we get the reduced system $H = \bigotimes_{k=1, k \neq 3}^{6} H^{A_k}$ and the reduced 5-qubits pure state $|\psi\rangle = \frac{1}{2}(|00000\rangle + |00111\rangle + |11000\rangle + |11111\rangle)$. (2) For two parties combinations, let $J_k = \{A_{k_1}, A_{k_2}\}$, where $\{k_1, k_2\} \subset \{1, 2, 4, 5, 6\}$ with $k_1 < k_2$ and $k = 1, 2, \cdots, 10$. We can easily calculate that $\mathrm{rank}(\rho_{J_1}) = 1$ and $\mathrm{rank}(\rho_{J_k}) = 4(k = 2, 3, \cdots, 10)$ which means that $J_1 = \{A_1, A_2\}$ is the second fully entangled combination, so we have $I_2 = (A_1, A_2)$. Tracing out the parties A_1 and A_2, we get the reduced system $H = \bigotimes_{k=4}^{6} H^{A_k}$ and the reduced 3-qubits pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ that is the GHZ state. (3) The remained parties A_4, A_5 and A_6 are fully entangled, we get the third combination $I_3 = (A_4, A_5, A_6)$. So we have $EC(|\psi\rangle) = [(A_3), (A_1, A_2), (A_4, A_5, A_6)]$.

Now we can introduce the Combinatorial Entropy for any multipartite pure states by using EC and CEF defined above.

Definition 3. *Suppose that $|\psi\rangle$ is a n-partite pure state in $H = H^{A_1} \otimes H^{A_2} \otimes \cdots \otimes H^{A_n}$. $P = \{A_1, A_2, \cdots, A_n\}$ is the parties set. $EC\,(|\psi\rangle) = [(I_1), (I_2), \cdots, (I_r)]$, The Combinatorial Entropy of $|\psi\rangle$ can be defined as:*

$$
\begin{aligned}
CE\,(|\psi\rangle) &= \sum_{k=1}^{r} CEF_{I_k}\,(|\psi_k\rangle) \qquad (9)\\
&= -\sum_{k=1}^{r} \sum_{\emptyset \neq J_{k_i} \subsetneq I_k} \mathrm{tr}\left(\rho_{J_{k_i}}\,(|\psi_k\rangle) \log_2 \rho_{J_{k_i}}\,(|\psi_k\rangle)\right)
\end{aligned}
$$

where $|\psi_k\rangle$ is the reduced pure state by tracing out all parties in $\bar{I}_k$ which means that $\rho_{I_k}\,(|\psi\rangle) = |\psi_k\rangle\langle\psi_k|$.

Combining the properties of CEF and EC we can easily get the following nice properties for CE.

Property 3. *(1) CE is just CEF for fully entangled pure states and CE is the partial entropy of bipartite pure states when $n = 2$.*

(2) CE is nonnegative for any multipartite pure state. $CE = 0$ if and only if the pure state is separable.

(3) CE is invariant under LU.

(4) The expectation of CE does not increase under LOCC.

(5) CE is additive for tensor products of independent states which means that if $|\psi\rangle$ and $|\phi\rangle$ are two pure states, we have $CE\,(|\psi\rangle \otimes |\phi\rangle) = CE\,(|\psi\rangle) + CE\,(|\phi\rangle)$.

The fourth property can be easily proved by using the following lemma [20].

Lemma 2. *If a multipartite system is initially in a pure state $|\psi\rangle$, and is subjected to a sequence of LOCC operations resulting in a set of final pure states $|\phi_i\rangle$ with probabilities p_i, then for any subset I of the parties*

$$S_I(|\psi\rangle) \geq \sum_i p_i S_I|\phi_i\rangle. \tag{10}$$

Taking the n-cat state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes n}\rangle + |1^{\otimes n}\rangle)$ for example, we have $EC(|\psi\rangle) = [(A_1, A_2, \cdots, A_n)]$ because $|\psi\rangle$ is fully entangled and $CE(|\psi\rangle) = 2^{n-1} - 1$.

For another example, let $|\psi\rangle = |EPR\rangle \otimes |GHZ\rangle, |\phi\rangle = |GHZ\rangle \otimes |EPR\rangle$ where $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. We can easily calculate that $EC(|\psi\rangle) = [(A_1, A_2), (A_3, A_4, A_5)], EC(|\phi\rangle) = [(A_4, A_5), (A_1, A_2, A_3)]$ and $CE(|\psi\rangle) = CE(|\phi\rangle) = CE(|EPR\rangle) + CE(|GHZ\rangle) = 4$ (For complicated examples, we can calculate CE by programming). These two pure state $|\psi\rangle$ and $|\phi\rangle$ have the same CE but have different EC which means that we should use both CE and EC to describe two different entangled states sometimes.

The quantum states discussed above in this paper are pure. EC is easily calculated as we have already constructed an efficient algorithm which is ascribed to those valid separability criterions for pure states. And CE can be used to quantify the entanglement for any pure states as it possesses those nice properties which is attributed to that the partial entropy is a good entanglement measure for any bipartite pure states. The mixed quantum states are more complicated than pure states as which bear entanglement together with classical probabilistic correlations.

Thanks to the considerable efforts of many researchers, now we have a variety of separability criterions for mixed states [21] which usually manifest themselves as some inequalities satisfied by any separable state, and if these inequalities are violated then the state cannot be separable, thus ascertain entanglement, but most of them are not sufficient.

The much harder work is to quantify entanglement for any bipartite mixed states. Even for two qutrits there is no consensus on how to quantify entanglement. Most entanglement measures, such as I-concurrence [22, 23], require a global minimization over all bases [6] which makes it cumbersome to calculate for mixed states. Some significant work on finding the numerical and analytical lower bound of I-concurrence have been proposed in [24, 25, 26]. However, analytical and computable entanglement measure for any bipartite mixed states is not still known.

We should point out that if we have obtained valid separability criterions and good entanglement measures for any bipartite mixed states, we can extend these three definitions (CEF EC and CE) to multipartite mixed states by using the same process discussed in this paper. Unfortunately, these two questions are still open now.

To summarize, in this paper we put forward three useful concepts. We can use EC to do the qualitative analysis of entanglement for any multipartite pure states. EC is easily obtainable as we have already constructed an efficient algorithm. By using EC and CEF we define CE which can be used to quantify the entanglement for any multipartite pure states. Because of those nice properties CE is a good entanglement measure. Finally, we point out that these concepts (CEF, EC and CE) can also be extended to mixed states if we have separability criterions and entanglement measures for any bipartite mixed states.

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